Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
January 19, 2024

1. This exam is four hours long.
2. The exam is closed-book; cheat sheets and notes are also not allowed.
3. You may need a scientific calculator for this exam.
4. Work out all six problems.
5. Start solution of each problem on a new page.
6. Number all of your pages.
7. Sign your name on the following line and put the total number of pages.
8. Use this sheet as a cover-sheet for your papers.

NAME: ________________________________

No. of pages:___________

Problem 1.

Consider the Newton's method iteration applied to solve the equation $f(x) = 0$,

$$x_{n+1} = x_n - f(x_n)/f'(x_n).$$

Suppose that f is twice continuously differentiable, $f(x^*) = 0$, and $f'(x^*) \neq 0$. Use Taylor's theorem with a remainder term to show that if x_n converges to x^*, then

$$\lim_{n\to\infty} \frac{|x^* - x_{n+1}|}{(x^* - x_n)^2} = C < \infty$$

Derive an explicit formula for the constant C.

Problem 2.

Let $f \in C^1[a,b]$; that is, f is continuously differentiable function defined on the interval $[a,b]$, and let $\|f\|_\infty = \max_{[a,b]} |f(x)|$. Let $x_0, \ldots, x_n$ be pairwise distinct numbers. Show that for every $\epsilon > 0$ there exists a polynomial p such that

$$||f - p||_\infty < \epsilon,$$

and

$$p(x_k) = f(x_k),\ 0 \leq k \leq n,$$

(p is an interpolant of f).

Hint: Let p_n be any interpolating polynomial of f, and let

$$\omega(x) = \prod_{k=0}^{n} (x - x_k).$$

Apply the Weierstrass approximation theorem to a function $(f - p_n)/\omega$. For a suitable polynomial q, let $p = p_n + \omega q$.

Theorem 1 (Weierstrass)**.** *If g is a continuous function on the interval $[a,b]$ and if $\epsilon > 0$, then there is a polynomial q satisfying $\|g - q\|_\infty \leq \epsilon$.*

Problem 3.

Consider a function $F : [a,b] \to R$. List the sufficient conditions that guarantee existence of the unique fixed point x^* of function F. Assuming these conditions, prove that the fixed point iteration

$$x_{n+1} = F(x_n), \quad n = 0, 1, 2, \ldots,$$

converges to x^*.

Problem 4.

a) Show that if A is a positive semidefinite matrix and λ is any eigenvalue of A, then $\lambda \geq 0$.

b) Conversely, show that if A is an n by n symmetric real matrix, and the eigenvalues λ_1, λ_2, ..., λ_n are all nonnegative, then A is positive semidefinite.

Parts (a) and (b) together show that an equivalent definition of a positive semidefinite real and symmetric matrix is that its eigenvalues are all nonnegative.

Problem 5.

Let

$$\|x\|_\infty = \max_i |x_i|, \; x = (x_1, \dots, x_n)^T \in R^n,$$

and, for any matrix A, let

$$\|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}.$$

Prove the following statement:

Theorem 2. *For any square matrix $A = (a_{ij})$,*

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|.$$

Hint: First, bound the left hand side of the identity by its right hand side by considering $\|Ax\|_\infty$. Then prove the opposite inequality. To this end, find a vector y such that $\|y\|_\infty = 1$, and

$$\max_i \sum_j |a_{ij}| = \sum_j |a_{kj}| = (Ay)_k,$$

and use the inequality $\|Ay\|_\infty \leq \|A\|_\infty \|y\|_\infty$.

Problem 6.

a) A Householder reflector Q that maps a vector x to a vector y such that $y \neq x$ and $\|y\|_2 = \|x\|_2$ is $Q = I - \gamma u u^T$ for some scalar γ and a vector u. Give the formulas for γ and u. Show that Q is an orthogonal matrix.

b) Find a Householder reflector Q that maps vector $x = (5, 2, -4, 2)^T$ to a vector of the form $y = (-\tau, 0, 0, 0)^T$; that is, find the scalar γ and the vector u for the given x and y. Verify that $Qx = y$.