Date: December 9, 2017 **Name:**

PhD Preliminary Examination in Analysis

Department of Mathematics
New Mexico Tech

2017

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. Prove that f is integrable with respect to α on $[a, b]$, that is, $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

2. Suppose that a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the recursive relation
$$a_{n+2} = \frac{1}{2}\left(a_n + a_{n+1}\right), \qquad \text{for all} \quad n \geq 1.$$
Prove that $\{x_n\}_{n=1}^{\infty}$ is convergent and find its limit.

3. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$, $f(0) = 0$, and $f'(x)$ finite for each x in $(0, 1)$. Define a function $g : (0, 1) \to \mathbb{R}$ by $g(x) = \frac{f(x)}{x}$ for $x \in (0, 1)$. Prove that if f' is an increasing function on $(0, 1)$, then g is also increasing on $(0, 1)$.

4. Let $f : (0, 1) \to \mathbb{R}$ be a function. Assume that: a) f is nonnegative and has a finite third derivative f''' in the open interval $(0, 1)$, and b) there exists $a, b \in (0, 1)$ such that $f(a) = f(b) = 0$. Prove that there exists $c \in (0, 1)$ such that $f'''(c) = 0$.

5. Let f and g be analytic functions on a region A in the complex plane. Suppose that f is injective (one-to-one) and $f'(z) \neq 0$ for any $z \in A$. Let C be a simple (without self-intersections) closed contour in A oriented counterclockwise. Let z_0 be a point inside C. Show that
$$\oint_C \frac{dz}{2\pi i} \, \frac{g(z)}{f(z) - f(z_0)} = \frac{g(z_0)}{f'(z_0)} \, .$$

6. Let P be a polynomial. Let C be a sufficiently large circle centered at the origin and oriented counterclockwise. Show that the integral
$$I = \oint_C \frac{dz}{2\pi i} z \frac{P'(z)}{P(z)}$$
is equal to the sum of the roots of the polynomial P counted with multiplicities.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)| \leq K|z|$ for all $z \in \mathbb{C}$, with some real constant K. Show that there is a real constant C such that $f(z) = Cz$ for all $z \in \mathbb{C}$.

8. Let m, n be two positive real numbers such that $0 < m < n$. Show that
$$\int_0^{\infty} dx \frac{x^{m-1}}{1 + x^n} = \frac{\pi}{n} \frac{1}{\sin\left(\frac{m}{n}\pi\right)}$$